2. 1 Basic structures

- Sets
- Functions
- Sequences
- Sums

Sets

- Used to group objects together
- Objects of a set often have similar properties
  - all students enrolled at UC Merced
  - all students currently taking discrete mathematics
- A **set** is an unordered collection of objects
- The objects in a set are called the **elements** or **members** of the set
- A set is said to **contain** its elements

Notation

- $a \in A : a$ is an elemnet of the set A. $a \notin A :$ otherwise
- The set of all vowels in the English alphabet can be written as V={a, e, i, o, u}
- The set of odd positive integers less than 10 can be expressed by O={1, 3, 5, 7, 9}
- Nothing prevents a set from having seemingly unrelated elements, {a, 2, Fred, New Jersey}
- The set of positive integers<100: {1,2,3,…, 99}

Notation

- **Set builder**: characterize the elements by stating the property or properties
- The set O of all odd positive integers < 10:

  O={x|x is an odd positive integer < 10}

  or specify as

$$O = \{x \in Z^+ \mid x \text{ is odd and } x < 10\}$$

- The set of positive rational numbers

$$Q^+ = \{x \in R \mid x = p/q \text{ for some positive integers } p \text{ and } q\}$$

Notation

$N = \{1,2,3,\ldots\}$ the set of natural numbers

$Z = \{\ldots,-2, -1, 0, 1,\ldots\}$ the set of integers

$Z^{+} = \{1,2,3,\ldots\}$ the set of positive integers

$Q = \{p/q \mid p \in Z, q \in Z, \text{and } q \neq 0\}$ the set of rational numbers

R, the set of real numbers

- The set {N, Z, Q, R} is a set containing four elements, each of which is a set

Sets and operations

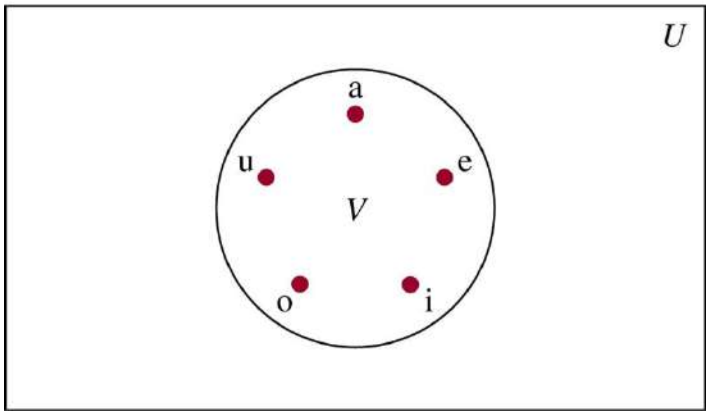
- A **datatype** or **type** is the name of a set,
- Together with a set of operations that can be performed on objects from that set
- **Boolean**: the name of the set {0,1} together with operations on one or more elements of this set such as AND, OR, and NOT

Sets

- Two sets are equal **if and only if** they have the same elements
- That is if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write A=B if A and B are equal sets
- The sets {1, 3, 5} and {3, 5, 1} are equal
- The sets {1, 3, 3, 3, 5, 5, 5, 5} is the same as {1, 3, 5} because the have the same elements

Venn diagram

- Rectangle: **Universal** set that contains all the objects
- Circle: sets
  - U: 26 letters of English alphabet
  - V: a set of vowels in the English alphabet



Empty set and singleton

- **Empty (null)** set: denoted by {} or Ø
- The set of positive integers that are greater than their squares is the null set
- **Singleton**: A set with one element
- A common mistake is to confuse Ø with {Ø}

Subset

- The set A is a subset of B **if and only if** every element of A is also an element of B
- Denote by $A \subseteq B$
- We see $A \subseteq B$ if and only if $\forall x(x \in A \rightarrow x \in B)$

Empty set and the set S itself

- Theorem: or every set S
  - (i) $\emptyset \subseteq S$, and
  - (ii) $S \subseteq S$
- Let S be a set, to show $\emptyset \subseteq S$, we need to show $\forall x(x \in \emptyset \rightarrow x \in S)$ is true.
- But $x \in \emptyset$ is always false, and thus the conditional statement is always true
- An example of vacuous proof
- (ii) is left as an exercise

Proper subset

- A is a **proper subset** of B: Emphasize that A is a subset of B but that A≠B, and write it as A⊂B

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

- One way to show that two sets have the same elements is to show that each set is a subset of the other, i.e., if A⊆B and B⊆A, then A=B

$$\forall x(x \in A \leftrightarrow x \in B)$$

Sets have other sets as members

- A={∅, {a}, {b}, {a,b}}
- B={x|x is a subset of the set {a, b}}
- Note that A=B and {a}∈A but a∉A
- Sets are used extensively in computing problem

Cardinality

- Let S be a set. If there are exactly n distinct elements in S where n is a non-negative integer
- S is a **finite** set
- $|S|=n$, n is the **cardinality** of S
  - Let A be the set of odd positive integers < 10, $|A|=5$
  - Let S be the set of letters in English alphabet, $|S|=26$
  - The null set has no elements, thus $|\emptyset|=0$

Infinite set and power set

- A set is said to be infinite if it is not finite
  - The set of positive integers is infinite
- Given a set S, the power set of S is the set of all subsets of the set S. The **power set** of S is denoted by P(S)
- The power set of {0,1,2}
  - $P(\{0,1,2\})=\{\emptyset,\{0\},\{1\},\{2\},\{0,1\},\{1,2\},\{0,2\},\{0,1,2\}\}$
  - Note the empty set and set itself are members of this set of subsets

- 2^n
- A={2,3,4,5}
- {{},{2},{3},{4},{5},{2,3},{2,4},{2,5},{3,4},{3,5},{4,5},{2,3,4},{2,3,5},{2,4,5},{3,4,5},{2,3,4,5}}

Example

- What is the power set of the empty set?
  - $P(\emptyset)=\{\emptyset\}$
- The set $\{\emptyset\}$ has exactly two subsets, i.e., $\emptyset$, and the set $\{\emptyset\}$. Thus $P(\{\emptyset\})=\{\emptyset,\{\emptyset\}\}$
- If a set has n elements, then its power set has 2^n elements

Cartesian product

- Sets are unordered
- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, and an as its nth element
- $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, .., n$

Ordered pairs

- 2-tupels are called ordered pairs
- (a, b) and (c, d) are equal if and only if a=c and b=d
- Note that (a, b) and (b, a) are not equal unless a=b

Cartesian product

- The **Cartesian product** of sets A and B, denoted by A x B, is the set of all ordered pairs (a,b), where a ∈ A and b ∈ B

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

- A: students of UC Merced, B: all courses offered at UC Merced
- A x B consists of all ordered pairs of (a, b), i.e., all possible enrollments of students at UC Merced

Example

- A={1, 2}, B={a, b, c}, What is A x B?
  - A x B=R = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}
- A subset R of the Cartesian product A x B is called a **relation**
- A={a, b, c} and B={0, 1, 2, 3}, R={(a, 0), (a, 1), (a,3), (b, 1), (b, 2), (c, 0), (c,3)} is a relation from A to B
- A x B $\neq$ B x A
  - B x A = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}

- $A=\{1,2\}$
- $B=\{a,b\}$
- $AxB=\{(1,a),(1,b),(2,a),(2,b)\}$
- $BxA=\{(a,1),(a,2),(b,1),(b,2)\}$

Cartesian product: general case

- Cartesian product of A_1, A_2, …, A_n, is denoted by A_1 x A_2 x … x A_n is the set of ordered n-tuples (a_1, a_2, …, a_n) where ai belongs to A_i for i=1, 2, …, n

$$A_1 \times A_2 \times \cdots A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{for } i = 1, 2, \ldots, n$$

- A={0,1}, B={1,2}, C={0,1,2}

  A x B x C={{0, 1, 0},{0, 1, 1}, {0, 1, 2}, {0, 2, 0}, {0, 2, 1}, {0, 2, 2}, {1, 1, 0}, {1, 1, 1}, {1, 1, 2}, {1, 2, 0}, {1, 2, 1}, {1, 2, 2}}

Using set notation with quantifiers

- $\forall x \in S(P(x))$ denotes the universal quantification P(x) over all elements in the set S
- Shorthand for $\forall x(x \in S \rightarrow (P(x))$
- $\exists x S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$
- What do they man? $\forall x \in R(x^2 \geq 0), \exists x \in Z(x^2 = 1)$
  – The square of every real number is non-negative
  – There is an integer whose square is 1

Truth sets of quantifiers

- Predicate P, and a domain D, the truth set of P is the set of elements x in D for which P(x) is true, denote by $\{x \in D | P(x)\}$
- $P(x)$ is $|x|=1$, $Q(x)$ is $x^2=2$, and $R(x)$ is $|x|=x$ and the domain is the set of integers
  - Truth set of P, $\{x \in Z | |x|=1\}$, i.e., the truth set of P is $\{-1,1\}$
  - Truth set of Q, $\{x \in Z | x^2=2\}$, i.e., the truth set is $\emptyset$
  - Truth set of R, $\{x \in Z | |x|=x\}$, i.e., the truth set is N

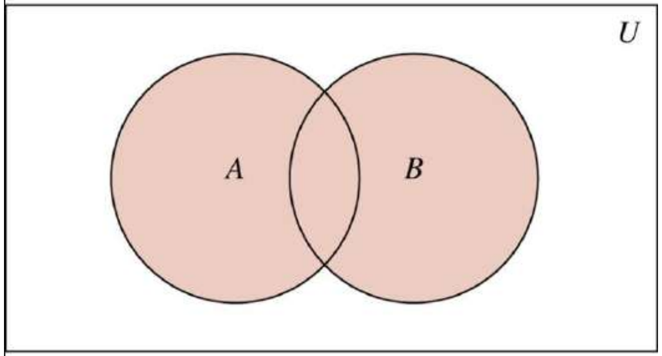
Example

- $\forall x P(x)$ is true over the domain U if and only if P is the set U
- $\exists x P(x)$ is true over the domain U if and only if P is non-empty

2.2 Set operations

- **Union**: the set that contains those elements that are either in A or in B, or in both

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



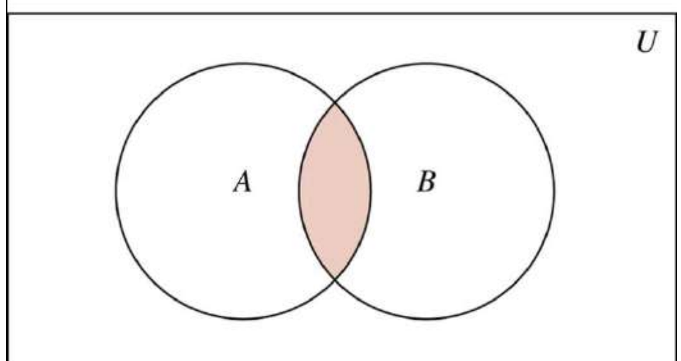
A *B* is shaded.

- A={1,3,5}, B={1,2,3}, AUB={1,2,3,5}

Intersection

- **Intersection**: the set containing the elements in both A and B

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

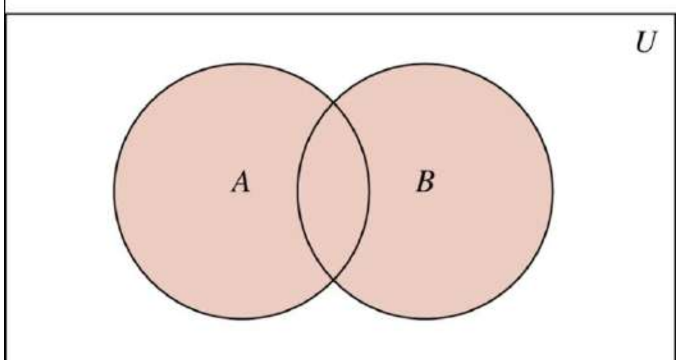


A   *B* is shaded.

- A={1,3,5}, B={1,2,3}, A⋂B={1,3}

Disjoint set

- Two sets are **disjoint** if their intersection is ∅
- A={1,3}, B={2,4}, A and B are disjoint
- Cardinality: $|A \cup B| = |A| + |B| - |A \cap B|$



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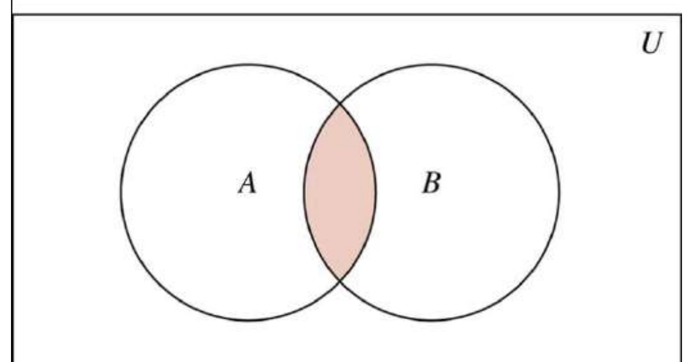
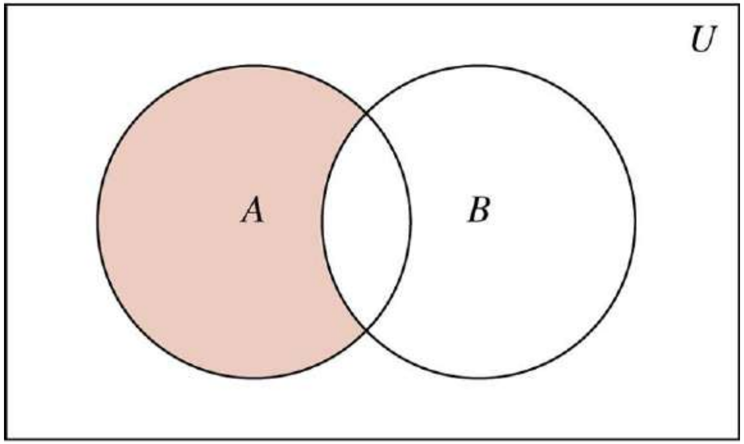


A *B* is shaded.

Difference and complement

- A-B: the set containing those elements in A but not in B $A-B=\{x \mid x \in A \wedge x \notin B\}$

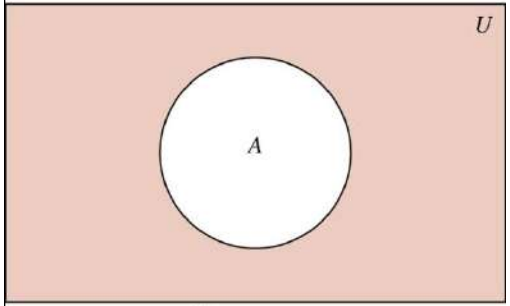


$A - B$ is shaded.

- A={1,3,5},B={1,2,3}, A-B={5} B-A={2}

Complement

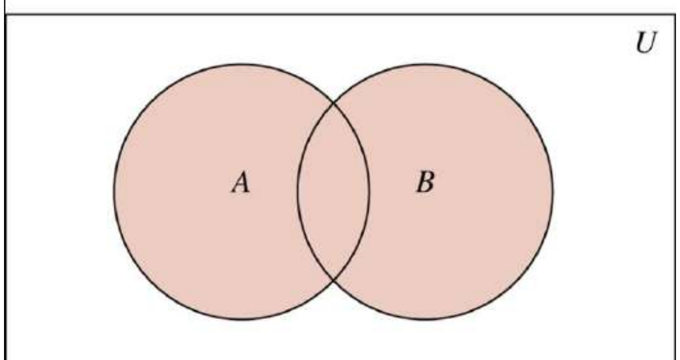
- Once the universal set U is specified, the complement of a set can be defined
- **Complement** of A: $\overline{A} = \{x \mid x \notin A\}, \overline{A} = U - A$
- A-B is also called the complement of B with respect to A



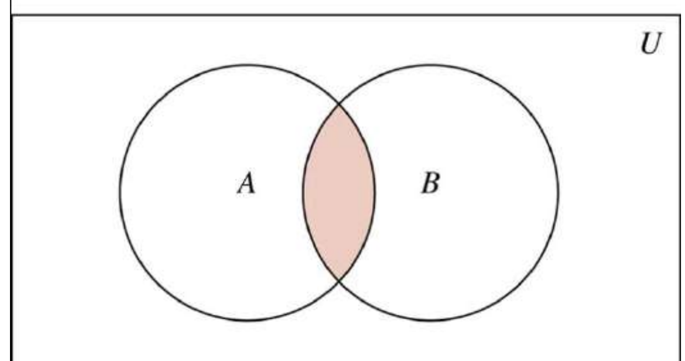
$\overline{A}$ is shaded.

Example

- A is the set of positive integers > 10 and the universal set is the set of all positive integers, then $\overline{A} = \{x \mid x \leq 10\} = \{1,2,3,4,5,6,7,8,9,10\}$
- A-B is also called the complement of B with respect to A



A *B* is shaded.



A *B* is shaded.

Set identities

TABLE 1 **Set Identities.**

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Example

- Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$
- (→): Suppose that $x \in \overline{A \cap B}$, by definition of complement and use De Morgan's law

$$\neg(x \in A \wedge x \in B)$$

$$\equiv (\neg(x \in A)) \vee (\neg(x \in B))$$

$$\equiv (x \notin A) \vee (x \notin B)$$

- By definition of complement $x \in \overline{A}$ or $x \in \overline{B}$
- By definition of union $x \in \overline{A} \cup \in \overline{B}$

Example

- (←): Suppose that $x \in \overline{A} \cup \overline{B}$
- By definition of union $x \in \overline{A} \vee x \in \overline{B}$
- By definition of complement $x \notin A \vee x \notin B$
- Thus $\neg(x \in A) \vee \neg(x \in B)$
- By De Morgan's law: $\neg(x \in A) \vee \neg(x \in B)$

$$\equiv \neg(x \in A \wedge x \in B)$$

$$\equiv \neg(x \in (A \cap B))$$

- By definition of complement, $x \in \overline{A \cap B}$

Builder notation

- Prove it with builder notation

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \quad \text{(def of complement)} \\
&= \{x \mid \neg(x \in (A \cap B))\} \quad \text{(def of not belong to)} \\
&= \{\mathrm{x} \mid \neg(x \in A \wedge x \in B)\} \quad \text{(def of intersection)} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \quad \text{(De Morgan's law)} \\
&= \{x \mid x \notin A \vee x \notin B\} \quad \text{(def of not belong to)} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \quad \text{(def of complement)} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} \quad \text{(def of union)} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Example

- Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- (→): Suppose that $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$. By definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$. Consequently, $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$
- By definition of intersection, it follows $x \in A \cap B$ or $x \in A \cap C$
- By definition of union, $x \in (A \cap B) \cup (A \cap C)$

Example

- (←): Suppose that $x \in (A \cap B) \cup (A \cap C)$
- By definition of union, $x \in A \cap B$ or $x \in A \cap C$
- By definition of intersection, $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$
- From this, we see $x \in A$, and $x \in B$ or $x \in C$
- By definition of union, $x \in A$ and $x \in B \cup C$
- By definition of intersection, $x \in A \cap (B \cup C)$

Membership table

TABLE 2 A Membership Table for the Distributive Property.

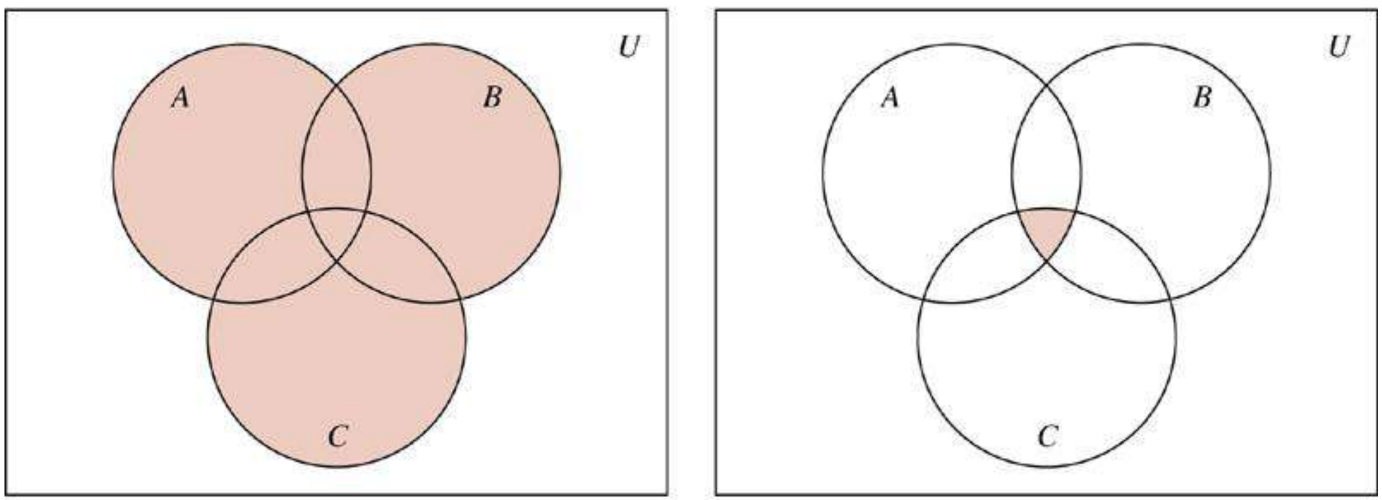
A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Example

- Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{B \cap C} \quad \text{(De Morgan's law)} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{(De Morgan's law)} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} \quad \text{(commutative law)} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} \quad \text{(commutative law)}
\end{aligned}
$$

Generalized union and intersection



(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

- A={0,2,4,6,8}, B={0,1,2,3,4}, C={0,3,6,9}
- $A \cup B \cup C = \{0,1,2,3,4,6,8,9\}$
- $A \cap B \cap C = \{0\}$

General case

- Union: $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$
- Intersection $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$
- Union: $A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i$
- Intersection: $A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i$
- Suppose A_i={1, 2, 3,…, i} for i=1,2,3,…

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1,2,3,\ldots,i\} = \{1,2,3,\ldots\} = Z^+$$

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1,2,3,\ldots,i\} = \{1\}$$

Computer representation of sets

- U={1,2,3,4,5,6,7,8,9,10}
- A={1,3,5,7,9} (odd integer ≤10),B={1,2,3,4,5} (integer ≤5)
- Represent A and B as 1010101010, and 1111100000
- Complement of A: 0101010101
- A⋂B: 1010101010^1111100000=1010100000

which corresponds to {1,3,5}